Pattern Matches in HOL: A New Representation and Improved Code Generation

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Overview

1. State of the Art
2. New Representation
3. Integration with CakeML
4. Examples
5. Conclusion

Overview

- pattern matching ubiquitous in functional programming
- pattern matching is a powerful technique
- it helps to write concise, readable definitions
- very handy and frequently used for interactive theorem proving in higher-order logic (HOL)
- however, it is not directly supported by HOL
- common representations
  - decision trees
  - sets of (conditional) equations

Decision Trees

- all major higher order logic systems use decision trees based on case-constants
  - **HOL 4**
  - Isabelle/HOL
  - ProofPower
  - HOL Light
  - Coq
  - ...
- algebraic datatype definitions introduce these case-constants
- the parser contains a pattern compilation algorithm that turns pattern matches in the input into decision trees
- pretty printers try to present these decision trees nicely

Decision Trees (Zip Example)

List Case Constant

```
list_case Nil n f = n      list_case (Cons y ys) n f = f y ys
```

Zipping Lists

```
zip l1 l2 = case (l1, l2) of
 | (Cons x l1', Cons y l2') =>
     Cons (x,y) (zip l1' l2')
 | _ => Nil
```

Zipping Lists (1)

```
zip l1 l2 =
  list_case l1 Nil (λ x l1'.
    list_case l2 Nil (λ y l2'.
      (Cons (x, y) (zip l1' l2'))))
```

- parser uses pattern compilation to produce a decision tree

Decision Trees (Zip Example)

List Case Constant

```
list_case Nil n f = n      list_case (Cons y ys) n f = f y ys
```

Zipping Lists

```
zip l1 l2 = case (l1, l2) of
 | (Cons x l1', Cons y l2') =>
     Cons (x,y) (zip l1' l2')
 | _ => Nil
```

Zipping Lists (2)

```
zip l1 l2 =
  list_case l2 Nil (λ y l2'.
    list_case l1 Nil (λ x l1'.
      (Cons (x, y) (zip l1' l2'))))
```

- parser uses pattern compilation to produce a decision tree
- exact result depends on pattern compilation heuristics

Decision Trees (Zip Example)

List Case Constant

```
list_case Nil n f = n      list_case (Cons y ys) n f = f y ys
```

Zipping Lists

```
zip l1 l2 = case (l1, l2) of
 | (Cons x l1', Cons y l2') =>
     Cons (x,y) (zip l1' l2')
 | _ => Nil
```

Zipping Lists (Pretty Printed)

```
zip l1 l2 = case (l1, l2) of
  | (Nil, _) => Nil
  | (Cons x l1', Nil) => Nil
  | (Cons x l1', Cons y l2') =>
       Cons (x,y) (zip l1' l2')
```

- parser uses pattern compilation to produce a decision tree
- exact result depends on pattern compilation heuristics
- pretty printer hides details

Decision Trees (RBT balancing)

Red-Black-Tree Constructors

```
Empty    Red    Black
```

Balancing (Okasaki, C.: Purely Functional Data Structures)

```
case (a,b) of
  (Red (Red a x b) y c,d) => Red (Black a x b) y (Black c n d)
| (Red a x (Red b y c),d) => Red (Black a x b) y (Black c n d)
| (a,Red (Red b y c) z d) => Red (Black a n b) y (Black c z d)
| (a,Red b y (Red c z d)) => Red (Black a n b) y (Black c z d)
| other => Black a n b
```

- optimal decision tree contains 57 cases
- naive compilation likely to produce more
- last case alone apears 36 times in result

Decision Trees (Discussion)

Issues

- big gap between user's view and internal representation
- pattern compilation trustworthy?
- non-trivial pretty printer trustworthy?
- size explosion and obfuscated structure
  - extracted code often complicated and slow
  - proofs lengthy and artificial

Function Definition Packages I

- various function definition packages provide an alternative to decision trees
- for top-level pattern-matches, they produce a set of (conditional) equations

Zip example revisited

```
|- ∀ l2. zip Nil l2 = Nil
|- ∀ l1. zip l1 Nil = Nil
|- ∀ x y l1’ l2’. zip (Cons x l1’, Cons y l2’) =
     Cons (x,y) (zip l1’ l2’)
```

Function Definition Packages II

Advantages

- no fancy pretty printer needed
- patterns not restricted
  - non-constructor functions
  - multiple occurences of bound variable
  - arithmetic expressions
  - ...
- conditional equations can represent guards

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Advantages

- no fancy pretty printer needed
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  - non-constructor functions
  - multiple occurences of bound variable
  - arithmetic expressions
  - ...
- conditional equations can represent guards

Limitations

- no precedence of rows
  - either add complicated guards
  - or have many equations
- only possible at top-level
- code extraction tricky

Overview

- we provide a powerful, direct representation of case-expressions in HOL 4
- available with HOL 4 in directory `examples/pattern_matches`
- idea very similar to concepts used by function definition packages
- main challange: engineering
  - decent parsing and pretty printing support
  - good automation (about 5000 lines of ML)
- CakeML (`cakeml.org`) has been extended to use new representation
- this representation allows to improve code generation

New Representation I

- a row is a function `r : `$\alpha \to \beta$` option`
- it matches a value `v : `α, if `r v` returns a value
- a case-expression searches for the first row matching
- if no row matches, it returns `PMATCH_INCOMPLETE`

Case Expressions

```
PMATCH v [] := PMATCH_INCOMPLETE := ARB
PMATCH v (r::rs) := case r v of
                      SOME result => result
                    | NONE => PMATCH v rs
```

New Representation II

- a row consists of a
  - a pattern `p` $: \gamma \to \alpha$
  - a guard `g` $: \gamma \to$ `bool`
  - a right-hand-side `r` $: \gamma \to \beta$
- a value `v` $: \alpha$ matches the row, if there setting `x` $: \gamma$ of the variables bound by `p` with `p x = v` and `g x`
- it then evalutes to `r` applied to the variable setting

Row Semantics

```
PMATCH_ROW p g r := λv.
  if (∃x. (p x = v) ∧ g x) then
    SOME (r (@x. (p x = v) ∧ g x))
  else
    NONE
```

Example List Membership

Raw Syntax (for MEM x l)

```
PMATCH l [
  PMATCH_ROW (λ(uv:unit). []) (λuv. T) (λuv. F);
  PMATCH_ROW (λ(y,ys). y::ys) (λ(y,ys). x = y) (λ(y,ys). T);
  PMATCH_ROW (λ(_0,ys). _0::ys) (λ(_0,ys). T) (λ(_0,ys). mem x ys)
]
```

Pretty Printer and Parser Syntax (for MEM x l)

```
CASE l OF [
  ||. [] ~> F;
  || (y,ys). y::ys when (x = y) ~> T;
  || ys. _::ys ~> mem x ys
]
```

Features

- guards
- patterns not restricted to constructors
- full control over variables
  - free variables in pattern
  - multiple occurences of bound variable
- non-injective patterns

Example

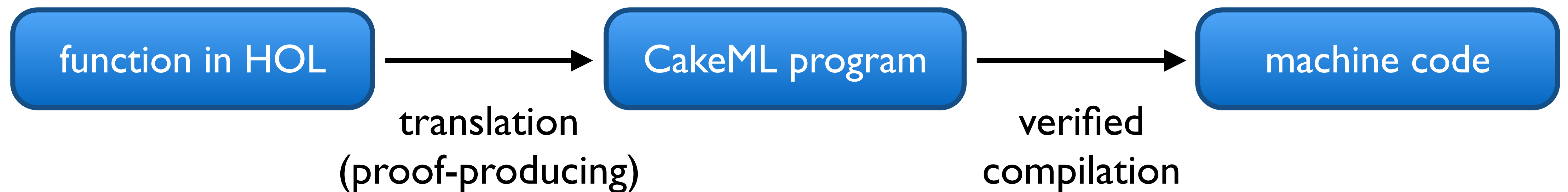
```
my_divmod n c :=
  CASE n OF [
    || (q, r). q * c + r when r < c ~> (q,r)
  ]
```

Tools

- conversions from and to decision trees
- congruence rules for automatic well-foundedness proofs
- (partial) evaluation and simplification tools
- removing by lifting to nearest boolean level
  - i.e. at top-level introducing equations like function definition packages
- powerful, extendable pattern compilation
- removal of redundant rows
- exhaustiveness checks
- removing guards / multiple variable bindings
- ...

CakeML

The CakeML project provides proof-producing code generation:



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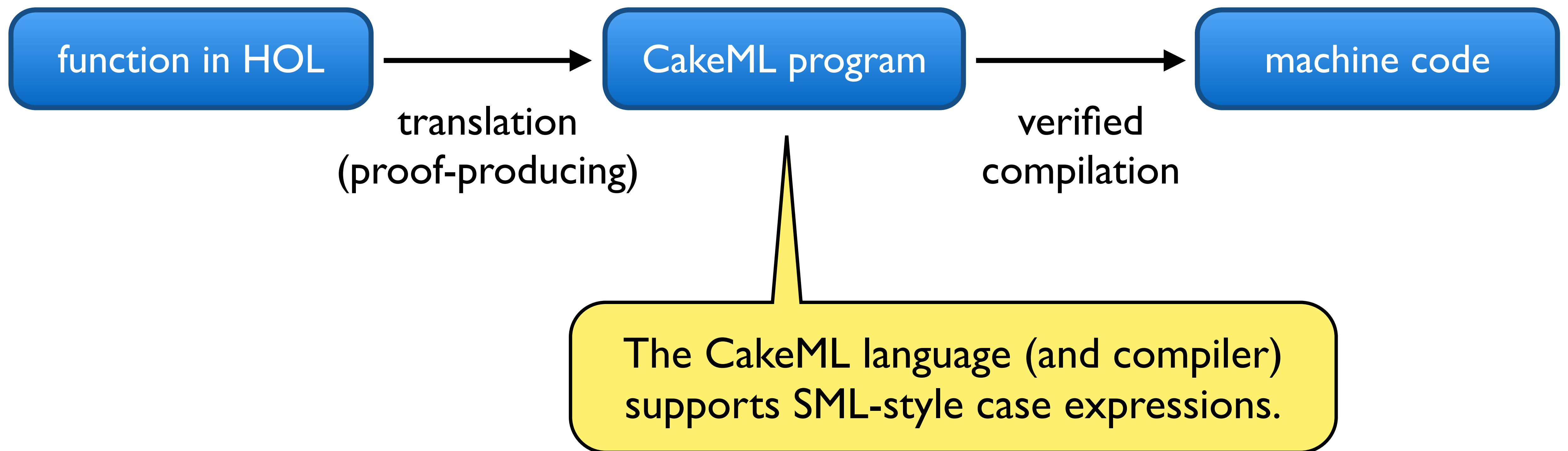

Code generation

User-input for a term that causes bad code generation:
(assuming a code generator without a fancy pattern compiler)

```
case (a,b) of
  (Red (Red a x b) y c,d) => Red (Black a x b) y (Black c n d)
| (Red a x (Red b y c),d) => Red (Black a x b) y (Black c n d)
| (a,Red (Red b y c) z d) => Red (Black a n b) y (Black c z d)
| (a,Red b y (Red c z d)) => Red (Black a n b) y (Black c z d)
| other => Black a n b
```

Without PMATCH, this turns into (at least) 57 rows in generated SML code.

State-of-the-art compilers can still produce optimal code from such suboptimal source code.

With PMATCH, a simple code generator can translate the above expression into 5 rows.

Even naive compilers (e.g. CakeML's) can produce reasonable code from this.

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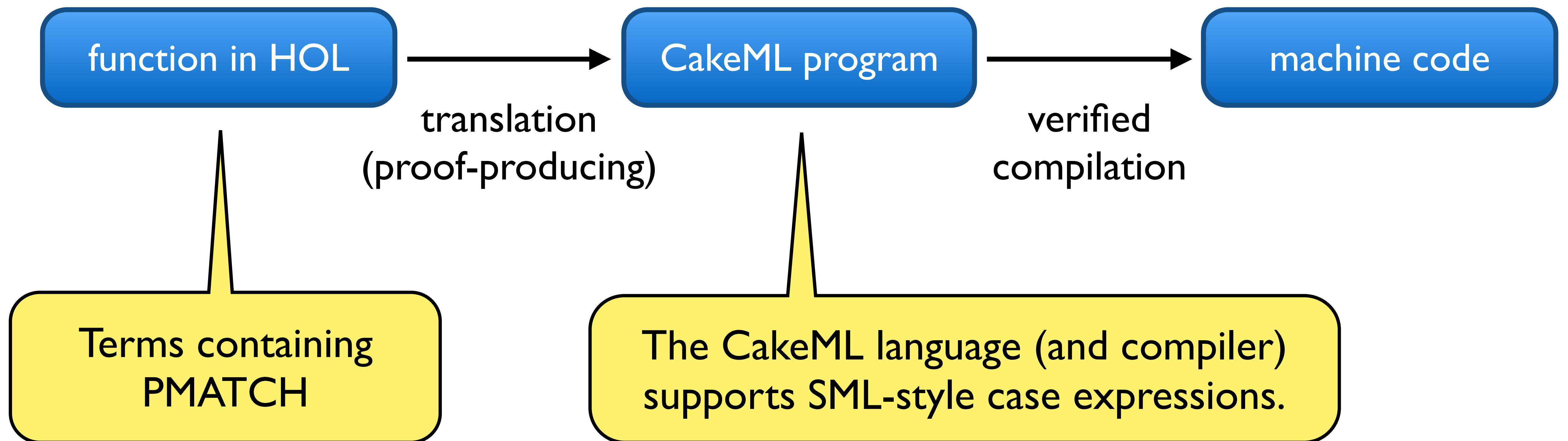

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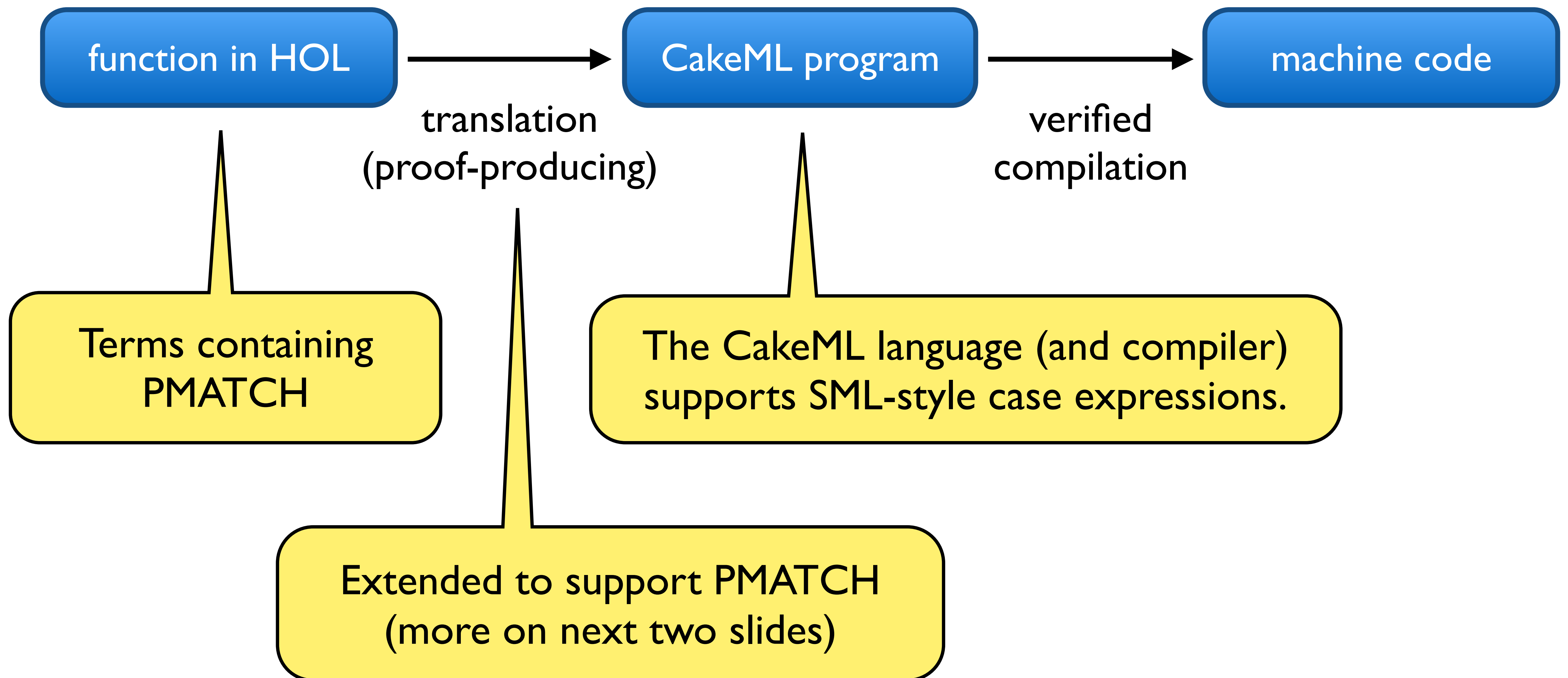
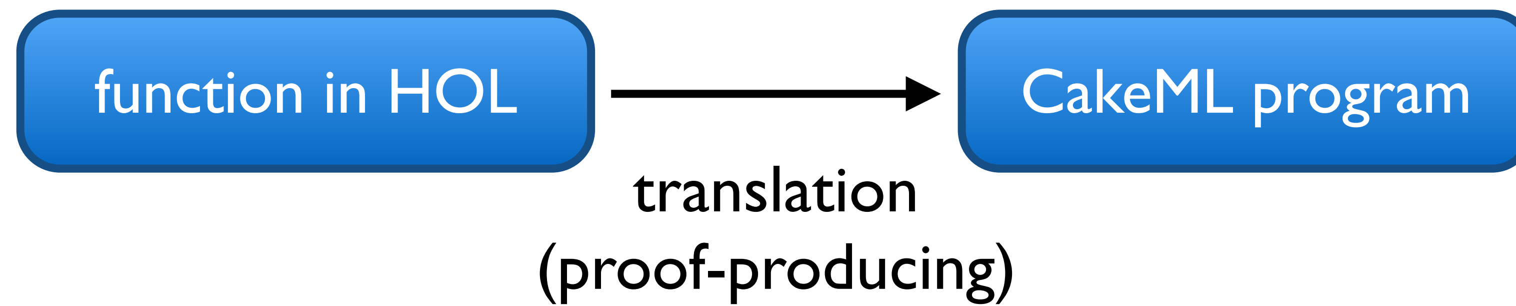


CakeML support for PMATCH



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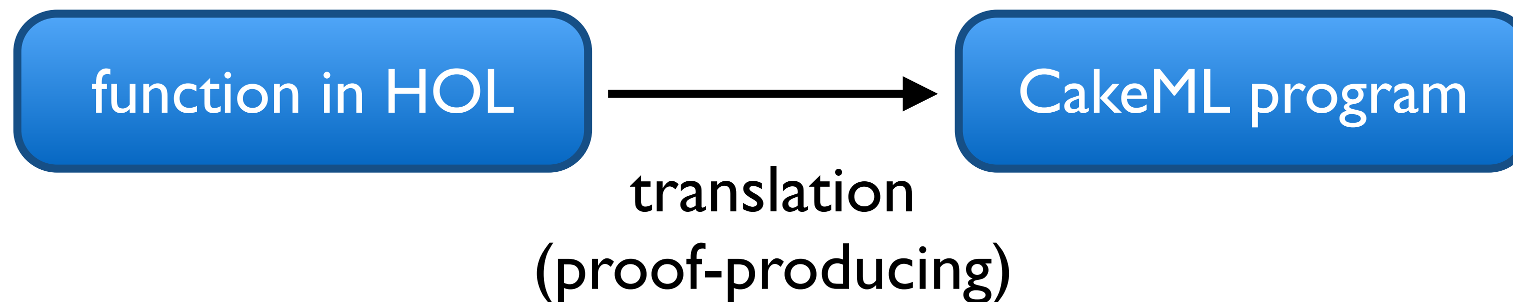
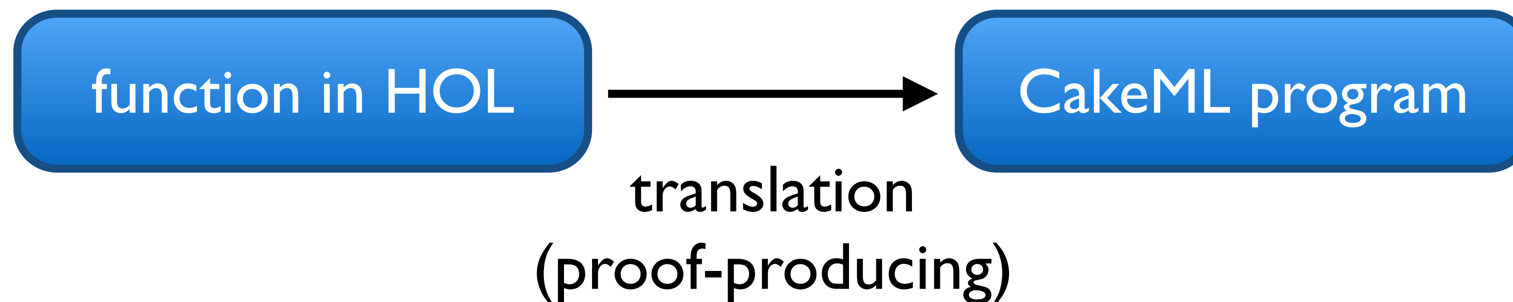


The translation derives theorems of the following form:

Eval cakeml_ast env (relation hol_term)

where Eval is: Eval ast env p = ∃v. (ast,env) ⇓ (Result v) ∧ p v

CakeML support for PMATCH



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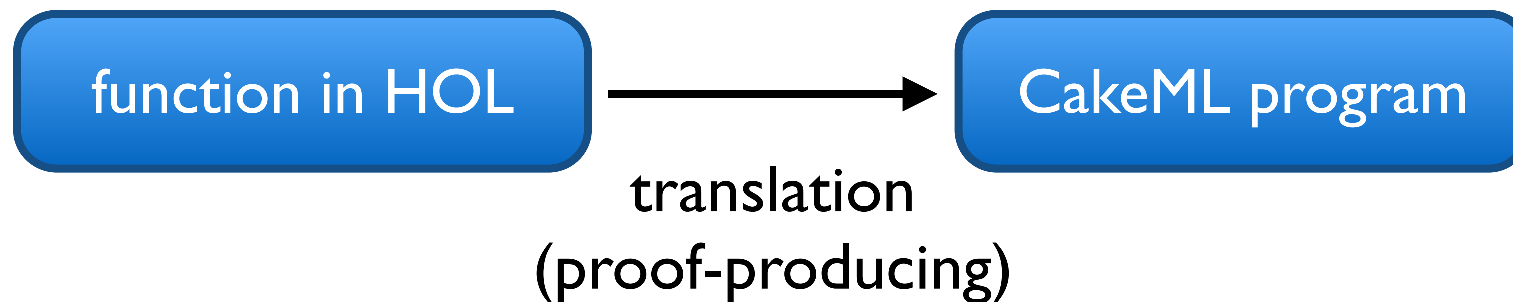
Eval cakeml_ast env (relation hol_term)

CakeML program

where Eval is: Eval ast env p = ∃v. (ast,env) ⇓ (Result v) ∧ p v

CakeML support for PMATCH



The translation derives theorems of the following form:

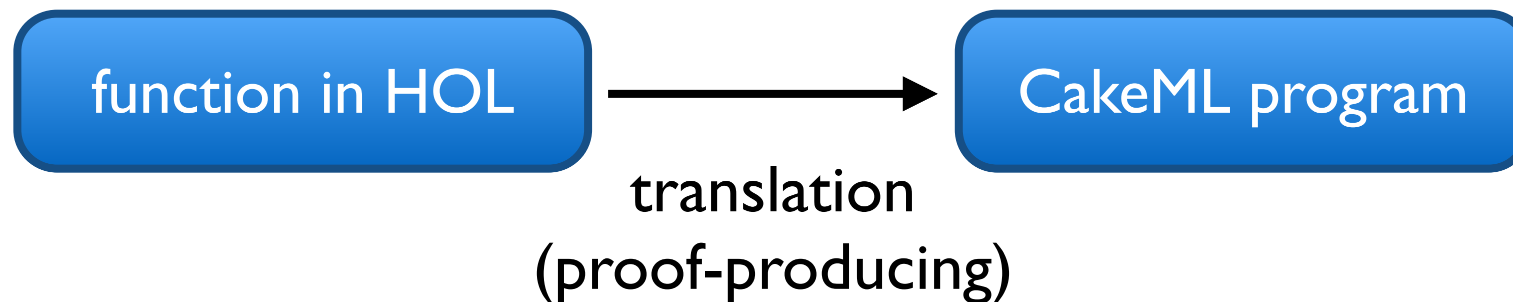
Eval cakeml_ast env (relation hol_term)

CakeML program

value relation

where Eval is: Eval ast env p = ∃v. (ast,env) ⇓ (Result v) ∧ p v

CakeML support for PMATCH



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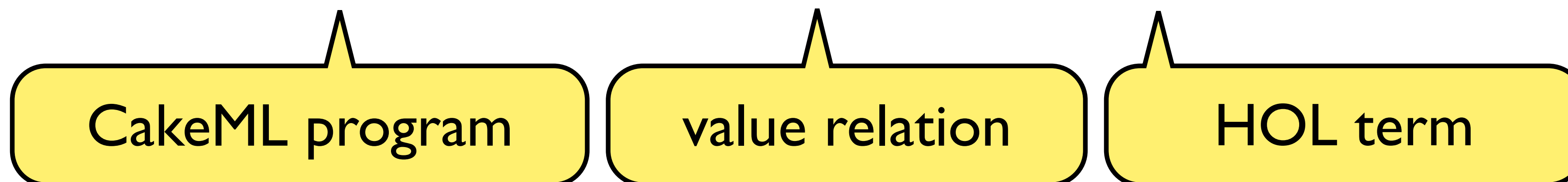

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CakeML support for PMATCH

The translation derives theorems of the following form:

Eval cakeml_ast env (relation hol_term)

CakeML program

value relation

HOL term

CakeML support for PMATCH

Implementation:

We supply two new theorems: one for translating PMATCH with no pattern rows, and one for adding a row to a PMATCH.

Custom automation written to apply the above theorems when PMATCH is to be translated.

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Eval cakeml_ast env (relation hol_term)

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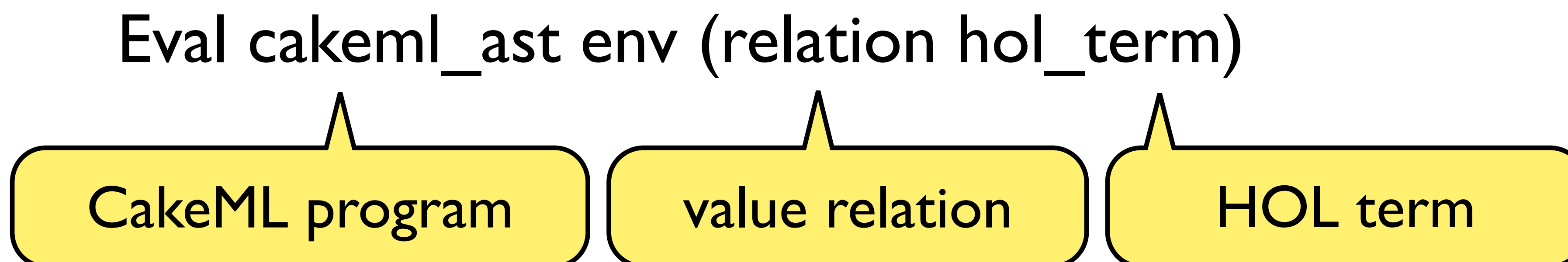

Experiments suggest that our implementation can translate all SML-style PMATCH terms of interest.

Partial Evaluation

```
|- CASE (a,Black b' m c') OF [
    || (a,x,b,y,c,d). (Red (Red a x b) y c,d) ~>
         (Red (Black a x b) y (Black c n d));
    || (a,x,b,y,c,d). (Red a x (Red b y c),d) ~>
         (Red (Black a x b) y (Black c n d));
    || (a,b,y,c,z,d). (a,Red (Red b y c) z d) ~>
         (Red (Black a n b) y (Black c z d));
    || (a,b,y,c,z,d). (a,Red b y (Red c z d)) ~>
         (Red (Black a n b) y (Black c z d));
    || other. other ~> (Black a n (Black b' m c'))
  ] =
  CASE a OF [
    || (a,x,b,y,c). Red (Red a x b) y c ~>
         (Red (Black a x b) y (Black c n (Black b' m c')));
    || (a,x,b,y,c). Red a x (Red b y c) ~>
         (Red (Black a x b) y (Black c n (Black b' m c')));
    ||. _ ~> (Black a n (Black b' m c'))
  ]
```

Removing Double Binds

```
|- REMOVE_DUPS l =
   CASE l OF [
     ||. [] ~> [];
     || (x,xs). x::x::xs ~> (REMOVE_DUPS (x::xs));
     || (x,xs). x::xs ~> (x::REMOVE_DUPS xs)
   ]
```

becomes by removing double binds

```
|- REMOVE_DUPS l =
   CASE l OF [
     ||. [] ~> [];
     || (x,xs,x_1). x::x_1::xs when (x_1 = x) ~>
           (REMOVE_DUPS (x::xs));
     || (x,xs). x::xs ~> (x::REMOVE_DUPS xs)
   ]
```

Removing Guards

```
|- REMOVE_DUPS l =
   CASE l OF [
     ||. [] ~> [];
     || (x,xs,x_1). x::x_1::xs when (x_1 = x) ~>
          (REMOVE_DUPS (x::xs));
     || (x,xs). x::xs ~> (x::REMOVE_DUPS xs)
   ]
```

becomes by removing guards

```
|- REMOVE_DUPS l =
   CASE l OF [
     ||. [] ~> [];
     || (x,xs,x_1). x::x_1::xs ~>
          if x_1 = x then REMOVE_DUPS (x::xs)
          else x::REMOVE_DUPS (x_1::xs);
     || (x,xs). x::xs ~> (x::REMOVE_DUPS xs)
   ]
```

Introducing Equations

```
|- (REMOVE_DUPS [] = []) ∧
   (∀x xs. REMOVE_DUPS (x::x::xs) = REMOVE_DUPS (x::xs)) ∧
   (∀x xs.
      (∀xs_1. xs ≠ x::xs_1) ⇒
      (REMOVE_DUPS (x::xs) = x::REMOVE_DUPS xs)) ∧
   (∀l. l ≠ [] ∧ (∀x xs. l ≠ x::xs) ⇒ (REMOVE_DUPS l = ARB))
```

```
|- (REMOVE_DUPS [] = []) ∧
   (∀x xs x_1. REMOVE_DUPS (x::x_1::xs) =
      if x_1 = x then REMOVE_DUPS (x::xs)
      else x::REMOVE_DUPS (x_1::xs)) ∧
   (∀x xs.
      (∀xs_1 x_1. xs ≠ x_1::xs_1) ⇒
      (REMOVE_DUPS (x::xs) = x::REMOVE_DUPS xs)) ∧
   (∀l. l ≠ [] ∧ (∀x xs. l ≠ x::xs) ⇒ (REMOVE_DUPS l = ARB))
```

Exhaustiveness

```
|- ∀l1 l2.
   ZIP_PART l1 l2 =
   CASE (l1,l2) OF [
     || (x,xs,y,ys). (x::xs,y::ys) ~> ((x,y)::ZIP_PART xs ys);
     ||. ([],[]) ~> []
   ]
```

becomes with automatically adding missing cases

```
|- ∀l1 l2.
   ZIP_PART l1 l2 =
   CASE (l1,l2) OF [
     || (x,xs,y,ys). (x::xs,y::ys) ~> ((x,y)::ZIP_PART xs ys);
     ||. ([],[]) ~> [];
     || (v4,v5). ([],v4::v5) ~> ARB;
     || (v2,v3). (v2::v3,[]) ~> ARB
   ]
```

We can prove exhaustiveness of this pattern-match automatically.

Pattern Compilation I

- one reason for blowup of RBT-balancing: 3 constructors
- only 2 cases needed: red or not
- our pattern compilation allows to do that
- resulting decision tree is about 20% of the size of original decision tree

```
val tree_red_CASE_def = Define `
  tree_red_CASE tr f_red f_else =
  tree_CASE tr (f_else Empty) f_red
    (1̑ n t2. f_else (Black t1 n t2))`

val red_cf = mk_constructorFamily (
  make_constructorList false [(``Red``, ["t1", "n", "t2"])],
  ``tree_red_CASE``,
  (* proof of basic properties *) ...);

val _ = pmatch_compile_db_register_constrFam red_cf;
```

Pattern Compilation II

```
|- ∀a n b.
  balance_black a n b = case (a,b) of (v,v_1) =>
    tree_red_CASE v
      (λt1 n_1 t2. tree_red_CASE t1
        (λt1_1 n_2 t2_1. Red (Black t1_1 n_2 t2_1) n_1 (Black t2 n v_1))
        (λx. tree_red_CASE t2
          (λt1_2 n_2_1 t2_1_1.
            Red (Black x n_1 t1_2) n_2_1 (Black t2_1_1 n v_1))
          (λx_1. tree_red_CASE v_1
            (λt1_3 n_2_2 t2_2.
              tree_red_CASE t1_3
                (λt1_1_1 n_3 t2_1_2.
                  Red (Black (Red x n_1 x_1) n t1_1_1) n_3
                    (Black t2_1_2 n_2_2 t2_2))
                (λx_2. tree_red_CASE t2_2
                  (λt1_4 n_3_1 t2_1_3.
                    Red (Black (Red x n_1 x_1) n x_2) n_2_2
                      (Black t1_4 n_3_1 t2_1_3))
                  (λx_3. Black a n b)))
            (λx_2_1. Black a n b))))
        (...)
```

Removing Redundant Rows

```
|- CASE ([],l) OF [
     ||.          ([], [])     ~> 0;
     || (x,xs). ([], x::xs) ~> 1;
     ||.          (_,  _)      ~> 2
   ] =
   CASE l OF [
     ||. []    ~> 0;
     ||. _::_ ~> 1;
     ||. _     ~> 2
   ]
```

```
|- CASE l OF [
     ||. []    ~> 0;
     ||. _::_ ~> 1;
     ||. _    ~> 2
   ] =
   CASE l OF [
     ||. []    ~> 0;
     ||. _::_ ~> 1
   ]
```

Conclusion

Summary

- we present a new representation for pattern matching in classical higher order logic
- implemented in HOL 4 (`http://hol-theorem-prover.org`)
- integration with CakeML (`http://cakeml.org`)
- tool support

Future Work

- more automation
- improved support for arithmetic expressions
- implement flattening
- ...

FireEye in Dresden

FireEye

- real-time threat protection against next generation cyber attacks
- customers: enterprises and governments worldwide

Formal Methods Team in Dresden, Germany

- hardening of trusted computing base
- verification of microhypervisor
- tools: Coq, static program analysis tools, ...
- currently 8 formal methods engineers
- expected to grow